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ROBUST VIBRATION CONTROL BASED ON READILY DETERMINED VARIABLES

Many dynamical systems such as cars, machine tools, planes or satellites suffer from vibrations caused by uncertain internal or external excitations. These mostly undesired vibrations affect either the comfort or the reliability of functional parts. One current approach to attenuate these vibrations is via so called active suspension elements. Mounted at appropriate places inside the systems or with respect to their environment, they are able to interchange or dissipate kinetic and potential energy in an effective way with moderate control effort. The effectiveness depends greatly on the control scheme applied to change damping and stiffness characteristics of the suspension elements. The control schemes however need very often information on the state variables in the mathematical model. On the other hand,mostly acceleration or speed of certain parts can be sensed reasonably and obtained with sufficient accuracy.

We propose an affordable control scheme which is solely based on these readily determined variables mentioned above. In addition, we only use control actions within a discrete set of possible values whichmakes them very easy to compute in real time. Furthermore, the number of control inputs (actuators)may be arbitrary, that is, the system may be mismatched. The scheme is based on Lyapunov stability theory and, provided that the bounds of the uncertainties are also a priori known, a stable attractor (ball of ultimate boundedness) of the structure can be computed.

The effectiveness and behavior of the control scheme is demonstrated on a simple model of an active car seat suspension to enhance the driving comfort.

1. Introduction

The class of systems which we shall take into consideration may be described by a dynamical system with a finite number of degrees of freedom. The structure has to contain "active" suspension elements. We call suspension or coupling elements "active" if they are adjustable with respect to their stiffness and damping behavior. Based on that model we assume that acontrol action is related to a change in these properties. The mathematical description of these kinds of structures is assumed to be of the form

$$\dot{x} = Ax + B(x) \cdot u + e, \; x(0) = x_0 \tag{1}$$

The linear part of the mathematical model of the structure to be controlled is defined by the constant and stable matrix $A \in IR^{n,n}$, where $n \in IN$ denotes the state space dimension. The control input matrix $B(x) := B_1(x) + B_2 \in IR^{n,m}$ may contain some constant part $B_2 \in IR^{n,m}$ and some part $B_1(x) \in IR^{n,m}$ which is linear with respect to $x.x \in IR^n$ represents the n state variables; $u \in IR^m$ represents the m control variables. Furthermore, we will assume that only $y := \dot{x}$ is detectable via some appropriate measurement device. The control variables have to be taken from the set:

$$\mathcal{U} := \{p: IR^n \to IR^m \mid p_j(y) \in \{u_{j,min}, 0, u_{j,max}\} \forall j = 1,\dots,m\} \tag{2}$$

where p is supposed to be piecewise continuous with respect to the measured and/or observed variable y. Without loss of generality and for the sake of convenience, we may assume that $u_{j,min} = -1$ and $u_{j,max} = +1$. The particular choice of a a control action $p_j(y)$ which either takes the minimum value $u_{j,min}$, 0, or the maximum value $u_{j,max}$, almost everywhere, is motivated by the control design presented in Leitmann et al. 1993. All uncertainties and nonlinearities of the system are modeled by some appropriate, at least piecewise differentiable function $e(t)$.

Its time derivative is assumedto be bounded: that is,

$$\|\dot{e}(t)\| \le \varepsilon \; \forall \, t \in IR_+ \tag{3}$$

holds for some properly chosen $\varepsilon \in IR_+$.

2. Control Design

For the controller design we ask for a feedback controller $u := p(y)$ which drives the measuredand/ or observed variable y towards a *ball of ultimate* boundedness $\mathcal{B}_\rho := \{\{\xi \in IR^n \mid \|\xi\| \le \rho\}$ for some properly chosen real number ρ > 0. As a first step towards this objective the time derivative of equation (1) leads to

$$\ddot{x} = A\dot{x} + \frac{d}{dt}[B(x) \cdot u] + \dot{e} \tag{4}$$

Furthermore, taking

$$\frac{d}{dt}[B(x)] = \frac{d}{dt}[B_1(x) + B_2] = B_1(\dot{x}) \tag{5}$$

into account - due to the linearity of B_1 with respect to x - leads to

$$\ddot{x} = A\dot{x} + B_1(\dot{x}) \cdot u + B(x) \cdot \dot{u} + \dot{e} \ . \tag{6}$$

And, since the components u_j of u take the constant values $u_{j,min}$, 0 or $u_{j,max}$ almost everywhere, that is on open subsets of IR^n except on measure zero sets of IR^n, the time derivative of $u = p(y)$ along any solution $t \mapsto y(t)$ of equation (6) is zero for all p belonging to u, provided chattering does not occur on the manifold

$$\Pi_j := \{ y \in IR^n \ | \ b_j(y) = 0 \} \tag{7}$$

for any j ∈ {1,..., n}. Hence, we obtain

$$\dot{y} = Ay + B_1(y) \cdot p(y) + \dot{e} \ . \tag{8}$$

This result enables us to return to the control design procedure introduced in Leitmann et al.(1993). There we ask for a feedback control function $p^* \in u$ which - for an arbitrary but fixed positive definite matrix $P \in IR^{n,n}$ - minimizes the "Lyapunov derivative"

$$L \quad (p) := y^T P[Ay + B_1(y)p + \dot{e}] = \tag{9}$$
$$= y^T P[Ay + \Sigma_{j=1}^m p_j B_1(y) i_j + \dot{e}]$$

with respect to $p \in$ U for every $(y; t) \in IR^n \times IR_+$. Here i_j denotes the unit vector with $i_i^T i_j = 0$ for $i \neq j$. In that case, the time derivative of the Lyapunov function candidate

$$V(y(t)) : \ = \frac{1}{2} y(t)^T Py(t) \tag{10}$$

will be as small as possible for any:

(i) response $t \mapsto y(t)$,
(ii) admissible uncertainty $\dot{e}$,
(iii) and time t,

and for all admissible choices of control $p(y(t))$. Equation (9) can be written as

$$L \quad (p \quad (y)) := a(y) + \sum_{j=1}^{m} p_j(y) b_j(y) + c(y,t) \tag{11}$$

with

$$a(y) := -\frac{1}{2} y^T Qy \qquad \text{where } Q := -(PA + A^T P)$$
$$b_j(y) := y^T PB(y) i_j \qquad \text{where } i_j i_k = \delta_{jk} \tag{12}$$

and

$$c(y,t) := y^T P\dot{e}(t) \tag{13}$$

Then, using the normalized control space u, we obtain

$$p_j^*(y) = \begin{cases} -1, & if \ \ b_j(y) > 0 \\ +1, & if \ \ b_j(y) < 0 \end{cases} \tag{14}$$

The performance of the controller may be enhanced additionally if we choose P appropriately. The smaller the Lyapunov derivative, the stronger the «tendency to the origin» of $t \mapsto y(t)$. Our objective in that case would be to strive for a highly negative value $a(y)$ in equation (11). That, on the other hand, can be done by choosing a suitable positive definite Q and solving the algebraic Lyapunov equation

$$Q = -(PA + A^T P) \tag{15}$$

for P. Since A is assumed to be stable, the matrix P is positive definite.

3. Stability

Using the control scheme developed in chapter 2 it is possible to determine a sufficient condition between parameters of the system and control design parameters for ultimate boundedness with respect to some ball $\mathcal{B}_\rho$ of any response $t \mapsto y(t)$. In order to analyze this situation we employ, for any given positive definite matrix $P \in IR^{n,n}$ the controller $p^*(y)$ (cf. equation (13)) in its corresponding Lyapunov derivative[1]. This leads to

$$L \ (p^*(y)) := -\frac{1}{2} y^T Qy + y^T PB_1(y) p^*(y) + y^T P\dot{e} \ .$$
$$= -\frac{1}{2} y^T Qy + y^T PB_1(y)(\Sigma_{j=1}^n p_j^*(y) i_j) + y^T P\dot{e}$$
$$= -\frac{1}{2} y^T Qy - \Sigma_{j=1}^n |b_j(y)| + y^T P\dot{e} \ . \tag{16}$$

Since $y^T Qy$ is a positive definite quadratic form, it is bounded by the minimum and maximum eigenvalue $\lambda_{min}(Q)$ and $\lambda_{max}(Q)$ of Q. That is,

$$L \ (p^*(y)) \leq -\frac{1}{2} \lambda_{min}(Q) \|y\|^2 - \tag{17}$$
$$- \sum_{j=1}^{m} |b_j(y)| + \|y\| \cdot \|P\| \cdot \|\dot{e}\| ,$$

or, neglecting the second term, employing inequality (3) and using the maximum eigenvalue $\lambda_{max}(P)$ of P

$$L \ (p^*(y)) \leq -\frac{1}{2} \lambda_{min}(Q) \|y\|^2 + \varepsilon \cdot \lambda_{max}(P) \|y\| \ . \tag{18}$$

Thus, we obtain

$$L \ (p^*(y)) \leq 0 \ \forall \ \|y\| > r := 2 \cdot \varepsilon \cdot \frac{\lambda_{max}(P)}{\lambda_{min}(Q)} . \tag{19}$$

The radius ρ of the *ball of ultimate boundedness* $\mathcal{B}_\rho := \{ \xi \in IR^n \, | \, \|\xi\| \leq \rho \}$ is therefore determined by

$$\rho = r \cdot \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \ . \tag{20}$$

Any response $t \mapsto y(t)$ which enters $\mathcal{B}_\rho$, say at $t = t^*$, remains in $\mathcal{B}_\rho$ for all $t > t^*$. It should be noted that $\mathcal{B}_\rho$, with radius given in (20), is the ball of ultimate boundedness for control actions $u_j = 0 \, \forall j$, since the terms $\sum_{j=1}^{m} |b_j(y)|$ were neglected in (18). These terms reduce the right hand size of (18), except at y where the $b_j(y) = 0$ for all j. In other words, the control acts to reduce the radius of $\mathcal{B}_\rho$ as well as the rate of convergence.

It should be noted that the control scheme (14) pertains to the open regions separated by switching manifolds"' (7). For $y(t)$ on a switching manifold Π_j, the terms dependent on du/dt, which is unbounded there, lead to Dirac delta changes in y and V in the ideal model under discussion. In practice, in a neighborhood of a switching point, rapid changes in y and V occur due to control with "high" gain over a "short" time interval. 1 In the stability analysis, we assumed that the rapid changes experienced by Lyapunov function $V(y(t))$ at a switch point of the control are sufficiently small, or even benign, so that they can be ignored. This seems to be borne out by the simulation results, but needs closer examination in general. We intend to investigate this phenomenon as well as that of possible chattering, in subsequent research.

4. Example: Active Suspension of a Car Seat

Fig 1 shows a simplifed model of an actively suspended seat in a car.

The car model consists of a mass M. Vertical vibrations caused by a rough street may be partially attenuated by shock absorbers (stiffness k_A and damping c_A). Nontheless, the driver may still be subjected to undesirable vibrations.

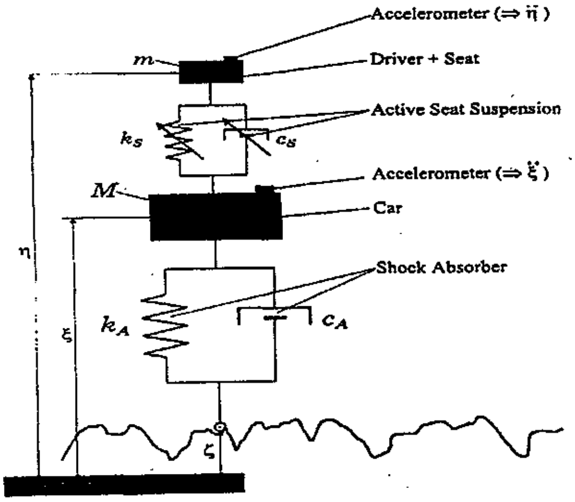

Figure 1: Model of an actively mounted seat inside a car

These vibrations, again, can be reduced by appropriately mounted car seat suspension elements. The elastic mounts are considered to be active with stiffness $k_S(u) = \alpha_k + \beta_k \cdot u$ and damping $c_S(u) = \alpha_c + \beta_c \cdot u$. u is the normalized and constrained control variable. That is, stiffness as well as damping can be varied only by changing the scalar variable u. The vertical displacement $\zeta(t)$ is considered to be unknown but possibly with a known bound. We assume that accelerations $\ddot{\xi}$ and $\ddot{\eta}$ are measured. The velocities $\dot{\xi}$ and $\dot{\eta}$ are either also measured, or at least estimated from their measured time derivatives. That leds to the following matrices for the model according to equation (1):

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_A + \alpha_k}{M}\right) & \frac{\alpha_k}{M} & -\left(\frac{c_A + \alpha_c}{M}\right) & \frac{\alpha_c}{M} \\ -\frac{\alpha_k}{m} & -\frac{\alpha_k}{m} & \frac{\alpha_c}{m} & -\frac{\alpha_c}{M} \end{bmatrix} \quad (21)$$

and

$$B(x) := B_1 \cdot x + B_2 , \quad (22)$$

where

$$B_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\beta_k}{M} & \frac{\beta_k}{M} & -\frac{\beta_c}{M} & \frac{\beta_c}{M} \\ \frac{\beta_k}{m} & -\frac{\beta_k}{m} & \frac{\beta_c}{m} & -\frac{\beta_c}{m} \end{bmatrix}, \; B_2 := \begin{bmatrix} 0 \\ 0 \\ -\frac{\beta_k \cdot \delta_S}{M} \\ \frac{\beta_k \cdot \delta_S}{m} \end{bmatrix} \quad (23)$$

The state vector is given by $x := (\xi, \eta, \dot{\xi}, \dot{\eta})^T$. The ground excitation leads to

$$e(t): = \begin{bmatrix} 0 \\ 0 \\ \frac{k_A}{M} \cdot (\zeta(t)) + \frac{c_A}{M} \cdot \dot{\zeta}(t) \\ 0 \end{bmatrix} . \quad (24)$$

Simulations for x and y are based on the original state equation (1) with control (14) which dependens on $y := \dot{x}$, namely

$$\dot{x} = Ax + B(x)p^*(y) + e(t) . \quad (25)$$

[1]It is possible to deal with uncertain state measurements in the manner discussed in Reithmeier et al. (2000), here setting $p_j(y) = 0$ in an appropriate neighborhood of Π_j for any jO{1,...,n}. In Reithmeier et al. (2000) a so-called fuzzy controller is used making the controller a continuous function of y, thereby also precluding chattering. However, here this can not be done in view of the discrete valuedness of admissible control.

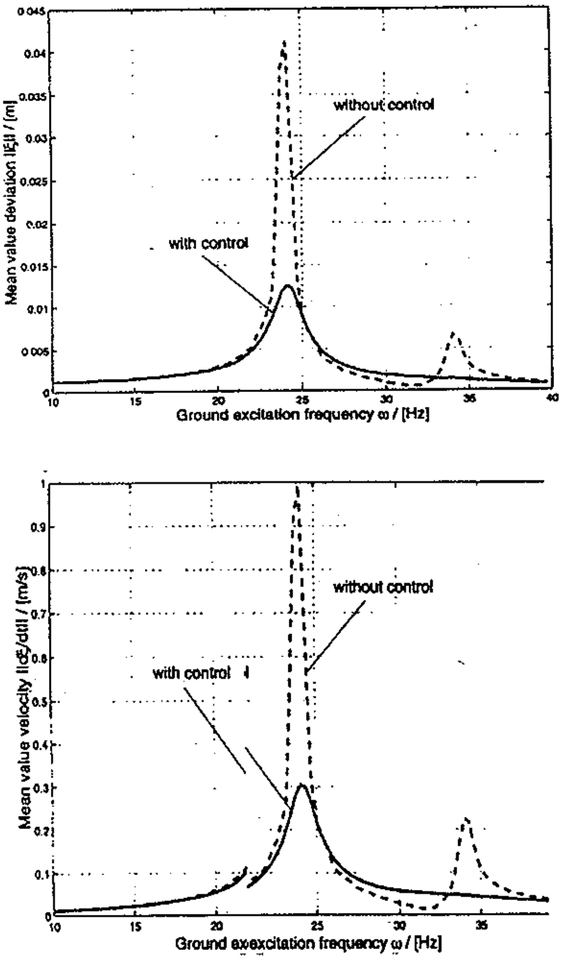

Figure 2: Mean values $|\xi|$ and $|\dot{\xi}|$ vs. excitation frequency ω

The simulation results (cf.Figures 2 and 3) are based on the parameters

g = 9.81 $[m/s^2]$ gravitational constant
m_C = 1500 $[kg]$ mass of the car
m_S = 100 $[kg]$ mass of the seat + driver
k_A = 10^6 $[N/m]$ shock absorber stiffness
c_A = $5\cdot10^2 [Ns/m]$ shock absorber damping
α_k = 10^5 $[N/m]$ (26)
β_k = $5\cdot10^4 [N/m]$
α_c = 60 $[Ns/m]$
β_c = 30 $[Ns/m]$

δ_A and δ_S are chosen in such a way that shock absorber elements are relaxed for $(\xi-\zeta)=\delta_A$ and the active seat suspension elements are relaxed for $(\eta-\xi)=\delta_S$. That leads to

$$\delta_S = \frac{m\cdot q}{\alpha_k} \qquad (27)$$

and

$$\delta_A = \frac{(m+M)\cdot q}{k_A} \qquad (28)$$

The ground excitation is taken to be harmonic, that is $\zeta(t) := \hat{\zeta}\cdot \sin(\omega\cdot t)$ with amplitude $\hat{\zeta} := 0.03[m]$ and a variable excitation frequency $10[Hz] \le \omega \le 40[Hz]$. In order to measure the effect of the employed controller, we integrate the system between time $t_0 = 0$ [s] and $t_2 = 10$ [s]. After a settling time $t_1 = 5$ [s] for the homogeneous parts of the state variables to be practically damped out, we compute the mean value

$$\|x_i\| := \sqrt{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} |x_i(t)|^2\, dt} \qquad (29)$$

for the variables x_i of $\boldsymbol{x}$ as functions of the excitation frequency ω. Without loss of generality, the numerical integration of (25) was started, with the initial condition $\boldsymbol{x}(0) = 0$. The values of $\boldsymbol{y}(t)$ may be directly obtained from (25). Figure 2 shows the results for the variables $x_1 = \xi$ and $x_3 = \dot{\xi}$, and Figure 3 for the variables $x_3 = \eta$ and $x_4 = \dot{\eta}$. Similar improvements in the stability behavior of the accelerations $\ddot{\xi}$ and $\ddot{\eta}$ take place. Interestingly, even though the control $\boldsymbol{p}^*(\boldsymbol{y})$ was designed to improve the stability behavior of $\boldsymbol{y}(t)$,

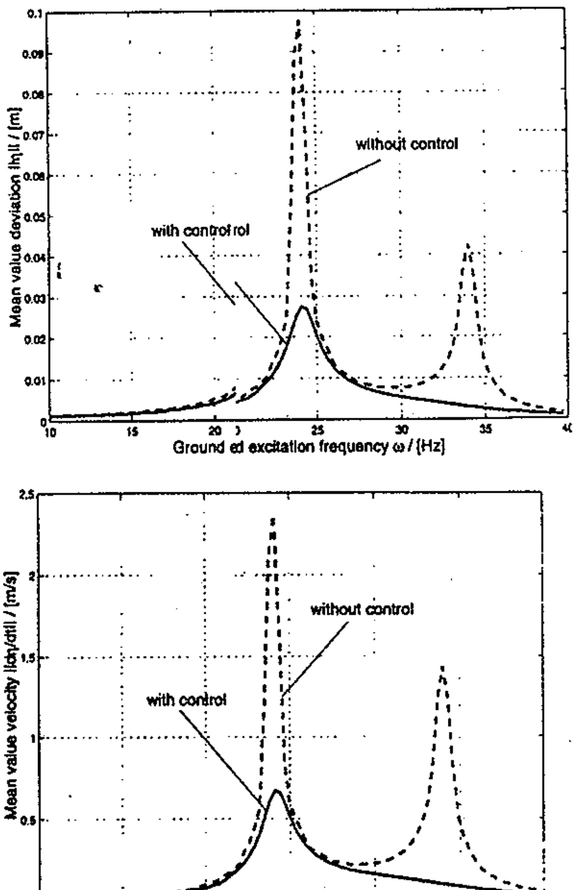

Figure 3: Mean values $|\eta|$ and $|\dot{\eta}|$ vs. excitation frequency ω

we see that the behavior of $x(t)$ is greatly improved as well; this is particularly so for the seat coordinates. Although there is a slight deterioration of ξ and $\dot{\xi}$ and between $\omega = 25$ [Hz] and $\omega = 33$ [Hz] of ξ and $\dot{\xi}$, the significant improvement of η and $\dot{\eta}$ prevails.

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References

1.M.Corless; G.Leitmann. Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamic Systems, IEEE Trans. Autom. Contr., 26, pp. 1139-1144, 1981.

2.R.E.Kalman; J.E.Bertram. Control Systems Analysis and Design via the Second Method of Lyapunov, J. Basic Eng., vol. 82, pp. 371 - 393, 1960.

3.J.M.Kelly; G.Leitmann; A.G.Soldatos. Robust Control of Base-Isolated Structures under Earthquake Excitation, J. Optimization Theory and Appl., vol. 52, no. 3, 1987.

4.G.Leitmann. Guaranteed Asymptotic Stability for Some Linear Systems With Bounded Uncertainties, Trans. ASME, vol. 101, pp. 212 - 216, 1979.

5.G.Leitmann; E.Reithmeier. Semiactive Control of a Vibrating System by Means of Electrorheological Fluids, Dynamics and Control, vol. 3, no. 1, 1993.

6.G.Leitmann. Semiactive Control for Vibration Attenuation, Journal of Intelligent Material Systems and Structures, vol. 5, pp. 841- 846, 1994.

7.E.Reithmeier; G.Leitmann. Semiaktive Regelung zur Amplitudenunterdr. uckung von Schwingungssystemen unter Einsatz elektrorheologischer Fl. ussigkeiten Archive of Applied Mechanics, 63, pp. 130-149, Springer Verlag, 1993.

8.E.Reithmeier; G.Leitmann. Robust Control of Seismic Structures Employing Active Suspension Elements. Advances in Structural Control. J. Rodellar, A. H. Barbat, F. Casciati (editors) . CIMNE, Barcelona. pp. 87-102, 1999.

9.E.Reithmeier; G.Leitmann. Structural Vibration Control. Journal of the Franklin Institute, 338 (2 - 3), pp. 203 - 223, 2001.

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