2 POLE PLACEMENT VIA LYAPUNOV FOR CONSTRAINED CONTROL OF MISMATCHED SYSTEMS

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1 Introduction

The main objective discussed in [5] is the suppression of undesired noise or vibrations in dynamical systems which are modelled by an o.d.e. of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{e}(\boldsymbol{x}, t) \quad \text{with} \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n. \tag{1}$$

The variable $\boldsymbol{x} \in \mathbb{R}^n$ describes the state of the system. $\boldsymbol{A} \in \mathbb{R}^{n,n}$ is assumed to be a constant and stable system matrix. $\boldsymbol{u} \in \mathbb{R}^m$ with $u_k \in [-1, +1]$ for $k = 1, \dots, m$ is the constrained control input. The input matrix $\boldsymbol{B} \in C^0[\mathbb{R}^n, \mathbb{R}^m]$ may be state-dependent. The *Caratheodory function* $\boldsymbol{e} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ models time-dependence, additional nonlinearities, and unknown disturbances. However, it is assumed to be uniformly bounded; that is, $\|\boldsymbol{e}(\boldsymbol{x}, t)\| \leq \eta$ for all $(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R}$, where the constant $\eta \in (0, \infty)$ may be unknown.

In [5, 6] the authors showed that the control design method based on Lyapunov stability theory leads to a unique control function

$$u_k = p_k(\boldsymbol{x}) = -\text{sgn}[b_k(\boldsymbol{x})], \quad k \in \{1, \dots, m\} \tag{2}$$

that minimizes the time derivative $\dot{V}(\boldsymbol{x}(t))$ of any given Lyapunov function candidate $V \in C^1[\mathbb{R}^n, \mathbb{R}]$ along any trajectory $t \mapsto \boldsymbol{x}(t)$ which satisfies equation (1) with control (2). The indicator function b_k in that case is given by

$$b_k(\boldsymbol{x}) = \sum_{j=1}^{n} \left[\frac{\partial V}{\partial x_j}(\boldsymbol{x}) \cdot B_{jk}(\boldsymbol{x}) \right] \quad \text{where} \quad B_{jk}(\boldsymbol{x}) := \boldsymbol{e}_j^T \boldsymbol{B}(\boldsymbol{x}) \tilde{\boldsymbol{e}}_k. \tag{3}$$

$$(\boldsymbol{e}_i^T \boldsymbol{e}_j = \delta_{ij},\ \tilde{\boldsymbol{e}}_i^T \tilde{\boldsymbol{e}}_j = \delta_{ij},\ \boldsymbol{e}_i \in \mathbb{R}^n,\ \tilde{\boldsymbol{e}}_i \in \mathbb{R}^m).$$

As shown in [6], the commonly used Lyapunov function candidate $V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$ with $\boldsymbol{P} \in \mathbb{R}^{n,n}$ and $\boldsymbol{P} > 0$ is actually a Lyapunov function if we determine $\boldsymbol{P}$ via the algebraic Lyapunov equation $\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} = -\boldsymbol{Q}$ where $\boldsymbol{Q} \in \mathbb{R}^{n,n}$ is any given positive definite matrix. In that case, the radius ρ of the ball of ultimate boundedness is given by

$$\rho = 2 \cdot \eta \cdot \frac{\lambda_{max}(\boldsymbol{P})}{\lambda_{min}(\boldsymbol{Q})} \cdot \sqrt{\frac{\lambda_{max}(\boldsymbol{P})}{\lambda_{min}(\boldsymbol{Q})}}. \tag{4}$$

$\lambda_{max}(\boldsymbol{P})$ and $\lambda_{min}(\boldsymbol{P})$ denote the maximum and minimum eigenvalues of $\boldsymbol{P} > 0$. The same holds for $\boldsymbol{Q} > 0$. At this point it should be noted that minimization of the Lyapunov derivative

$$\mathcal{L}_{(x,t)}[\boldsymbol{u}] := \boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{Ax} + \boldsymbol{Bu} + \boldsymbol{e}(\boldsymbol{x}, t)] \tag{5}$$

has also a geometric interpretation in $\mathbb{R}^n$: Since $\boldsymbol{P} > 0$ there is an invertible $\boldsymbol{T} \in \mathbb{R}^{n,n}$ such that $\boldsymbol{P} = \boldsymbol{T}^T\boldsymbol{T}$ or, along a trajectory of (1),

$$\begin{aligned} \mathcal{L}_{(\mathbf{x},t)}[\mathbf{u}] &= x^T\boldsymbol{T}^T\boldsymbol{T}\dot{\boldsymbol{x}} \\ &= (\boldsymbol{Tx})^T\frac{d}{dt}(\boldsymbol{Tx}), \quad \left[\frac{d}{dt}(\boldsymbol{Tx}) = \boldsymbol{T}\dot{\boldsymbol{x}}\right] \\ &= \boldsymbol{y}^T\dot{\boldsymbol{y}}, \quad [\boldsymbol{y} := \boldsymbol{Tx}]. \end{aligned} \tag{6}$$

Thus, the proposed control (2) minimizes the inner product between $\boldsymbol{y}(t)$ and $\dot{\boldsymbol{y}}(t)$; in other words, $\dot{\boldsymbol{y}}(t)$ points as closely as possible towards the origin. Of course, this tendency and hence the system performance depends on the choice of $\boldsymbol{T}$ or $\boldsymbol{P}$, respectively. In Section 3, in order to illustrate further the "optimality" of the proposed control (2), we employ a $\boldsymbol{T}$ such that

$$\boldsymbol{TAT}^{-1} = \boldsymbol{\Omega}, \tag{7}$$

where the real matrix $\boldsymbol{\Omega}$ is defined by

$$\boldsymbol{\Omega} := \left[\begin{array}{ccc|ccc} \begin{pmatrix} -\delta_1 & -\omega_1 \\ \omega_1 & -\delta_1 \end{pmatrix} & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & \begin{pmatrix} -\delta_{n_c} & -\omega_{n_c} \\ \omega_{n_c} & -\delta_{n_c} \end{pmatrix} & & & \\ \hline & & & -\mu_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -\mu_{n_r} \end{array}\right] \tag{8}$$

and

- $\lambda_{2k-1} = -\delta_k + i\omega_k$ and $\lambda_{2k} = -\delta_k - i\omega_k$ are the complex eigenvalues of $\boldsymbol{A}$
- $\mu_1, \ldots, \mu_{n_r}$ are the real eigenvalues of $\boldsymbol{A}$.

Furthermore, because of

$$
\begin{aligned}
-Q &= PA + A^T P \\
&= T^T TA + A^T T^T T \\
&= T^T (TAT^{-1}) T + T^T (T^T)^{-1} A^T T^T T \\
&= T^T \Omega T + T^T (TAT^{-1})^T T, && (\Omega = TAT^{-1}) \\
&= T^T (\Omega + \Omega^T) T \\
&= -T^T \Delta T, && (\Delta := -(\Omega + \Omega^T)) \\
&= -\left(\sqrt{\Delta} T\right)^T \left(\sqrt{\Delta} T\right) \\
&= -R^T R, && \left(R := \sqrt{\Delta} T\right)
\end{aligned}
\tag{9}
$$

or

$$Q = R^T R \tag{10}$$

respectively, this T defines a Lyapunov function, and (cf. [6]) the radius ρ of the ball of ultimate boundedness is given by

$$\rho = \frac{\eta}{\delta_{max}} \cdot \sqrt{\frac{\lambda_{max}(T^T T)}{\lambda_{min}(T^T T)}} \tag{11}$$

with

$$\delta_{max} := \max\{\delta_1, \ldots, \delta_{n_c}, \mu_1, \ldots, \mu_{n_r}\} \tag{12}$$

2 Lyapunov Approach as Limit Case of a Linear Constrained Control

The objective of this section is to show that for $B(x) = B = const.$ the controller design discussed in Section 1 and in [5, 6] is a limit case of pole placement within the set of all *admissible* linear constrained feedback controllers. The attribute *admissible* denotes the restriction of u to the set of constrained linear controllers

$$\mathcal{U}_a := \left\{ u \in C^0[\mathbb{R}^n, \mathbb{R}^m] \mid u_k(x) = -\mathrm{sat}(\tilde{e}_k^T K x);\ K \in \mathbb{R}^{m,n};\ k \in \{1, \ldots, m\} \right\} \tag{13}$$

with

$$\mathrm{sat}(f(x)) := \begin{cases} +1 & \text{if } f(x) < -1 \\ f(x) & \text{if } -1 \le f(x) \le 1; \quad f \in C^0[\mathbb{R}^n, \mathbb{R}] \\ -1 & \text{if } f(x) > 1 \end{cases} \tag{14}$$

Of course, the controller $\boldsymbol{p}(\boldsymbol{x})$ described in Section 1 does not belong to $\mathcal{U}_a$. However, there exists a continuous parameter deformation

$$\begin{aligned} \tilde{p}_k(\boldsymbol{x},\cdot): \quad &]0,\frac{\pi}{2}[\rightarrow \mathcal{U}_a \\ & \alpha \mapsto -\operatorname{sat}[\tan(\alpha)\tilde{\boldsymbol{e}}_k^T \boldsymbol{B}^T \boldsymbol{P}\boldsymbol{x}] \end{aligned} \tag{15}$$

$(k \in \{1,\ldots,m\})$ such that

$$p_k(\boldsymbol{x}) = \lim_{\alpha\rightarrow\pi/2} \tilde{p}_k(\boldsymbol{x},\alpha) \tag{16}$$

Suppose that a solution $t \mapsto \boldsymbol{x}(t)$ of (1) with $u_k = \tilde{p}_k(\boldsymbol{x},\alpha)$ is such that there is an interval $[t_1,t_2]$ during which not all components of $\boldsymbol{u}$ are saturated. Let

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_{II} \end{bmatrix}, \quad \boldsymbol{B} = [\boldsymbol{B}_I, \boldsymbol{B}_{II}] \Longrightarrow \boldsymbol{B}\boldsymbol{u} = \boldsymbol{B}_I\boldsymbol{u}_I + \boldsymbol{B}_{II}\boldsymbol{u}_{II} \tag{17}$$

where $\boldsymbol{u}_I$ denotes the unsaturated part of $\boldsymbol{u}$. Then the systems behavior on $[t_1,t_2]$ is governed by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_I\boldsymbol{u}_I + \boldsymbol{B}_{II}\boldsymbol{u}_{II} + \boldsymbol{e}(\boldsymbol{x},t) \\ &= (\boldsymbol{A} - \tan(\alpha)\boldsymbol{B}_I\boldsymbol{B}_I^T\boldsymbol{P})\boldsymbol{x} + \boldsymbol{B}_{II}\boldsymbol{u}_{II} + \boldsymbol{e}(\boldsymbol{x},t) \end{aligned} \tag{18}$$

That is, as in the Lyapunov approach of [5, 6], the controller design does not take care of the uncertain excitation $\boldsymbol{e}$, but rather improves the behavior of the *nominal* system and hence its behavior in the presence of disturbances.

With respect to (17), any arbitrary but fixed $\alpha \in]0,\frac{\pi}{2}[$ determines a pole distribution of the *nominal* system. These poles are the eigenvalues of

$$\tilde{\boldsymbol{A}}(\alpha) := \boldsymbol{A} - \tan(\alpha)\boldsymbol{B}_I\boldsymbol{B}_I^T\boldsymbol{P}. \tag{19}$$

In order to investigate the damping behavior of the controlled *nominal* system, we will take a look at one of the invariants of $\tilde{\boldsymbol{A}}$ which gives information about the real parts of the eigenvalues:

$$\begin{aligned} \tilde{\boldsymbol{A}}(\alpha) &= \operatorname{tr}(\boldsymbol{A} - \tan(\alpha)\boldsymbol{B}_I\boldsymbol{B}_I^T\boldsymbol{P}) \\ &= \operatorname{tr}(\boldsymbol{A}) - \tan(\alpha)\operatorname{tr}(\boldsymbol{B}_I\boldsymbol{B}_I^T\boldsymbol{T}^T\boldsymbol{T}) \\ &= \operatorname{tr}(\boldsymbol{A}) - \tan(\alpha)\operatorname{tr}(\boldsymbol{T}\boldsymbol{B}_I\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{T}^T) \\ &= -\sum_{k=1}^{n_c} 2\delta_k - \sum_{k=1}^{n_r} \mu_k - \tan(\alpha)\sum_{k=1}^{m_I} \sigma_k^2 \end{aligned} \tag{20}$$

where

$$\sigma_k^2 := \boldsymbol{e}_k^T \boldsymbol{T}\boldsymbol{B}_I\boldsymbol{B}_I^T\boldsymbol{T}^T\boldsymbol{e}_k^T. \tag{21}$$

That is, if $\alpha \to \frac{\pi}{2}$ then

$$\begin{aligned} &1)\quad \tilde{\boldsymbol{p}}(\boldsymbol{x},\alpha) \to \boldsymbol{p}(\boldsymbol{x}) \\ &2)\quad \sum_{k=1}^{n} \mathrm{Re}\{\lambda_k(\boldsymbol{A} - \tan(\alpha)\boldsymbol{B}_I\boldsymbol{B}_I^T\boldsymbol{P})\} \to -\infty \end{aligned} \tag{22}$$

In other words, the proposed Lyapunov approach leads to the strongest possible damping "on the average".

3 Test Example and Numerical Results

To illustrate the result in Section 2, we will employ a test example already used in [6], given by the system matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{m_1}[k_1+k_2] & \frac{k_2}{m_1} & -\frac{1}{m_1}[c_1+c_2] & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \tag{23}$$

with

$$m_i = 1[\mathrm{kg}],\ \ k_i = 1000[\mathrm{N/m}],\ \ c_i = 1[\mathrm{Ns/m}], \tag{24}$$

and control input matrix $\boldsymbol{B}$ and excitation $\boldsymbol{e}$ given by

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{m_2} \end{bmatrix},\quad \boldsymbol{e}(\boldsymbol{x},t) = \begin{bmatrix} 0 \\ 0 \\ -\frac{F}{m_1} \\ 0 \end{bmatrix} \tag{25}$$

with

- $F(t) = \bar{F}\sin(\nu \cdot t)$,
- $\bar{F} = 5[\mathrm{N}],\ \ \nu \in\ [10[1/\mathrm{s}], 80[1/\mathrm{s}]]$.

The feedback control employed is

$$\tilde{p}_k(\boldsymbol{x},\alpha) = -u_{max} \cdot \mathrm{sat}[\tan(\alpha)\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{e}}_k],\quad k \in \{1,\dots,4\}. \tag{26}$$

with

- $u_{max} = 2.5[\mathrm{N/m}]$.

That is, we consider a fairly simple example which, however, accounts for

- mismatched uncertainty,
- constrained excitation,
- and constrained control.

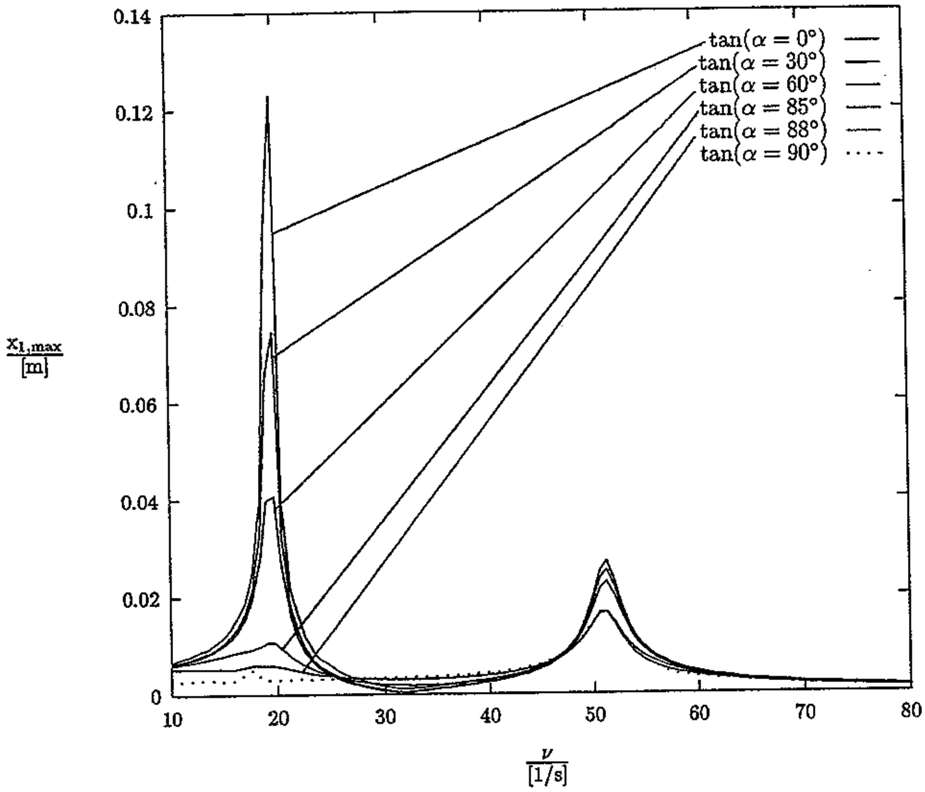

Figure 1 $x_{1,max}$ versus ν.

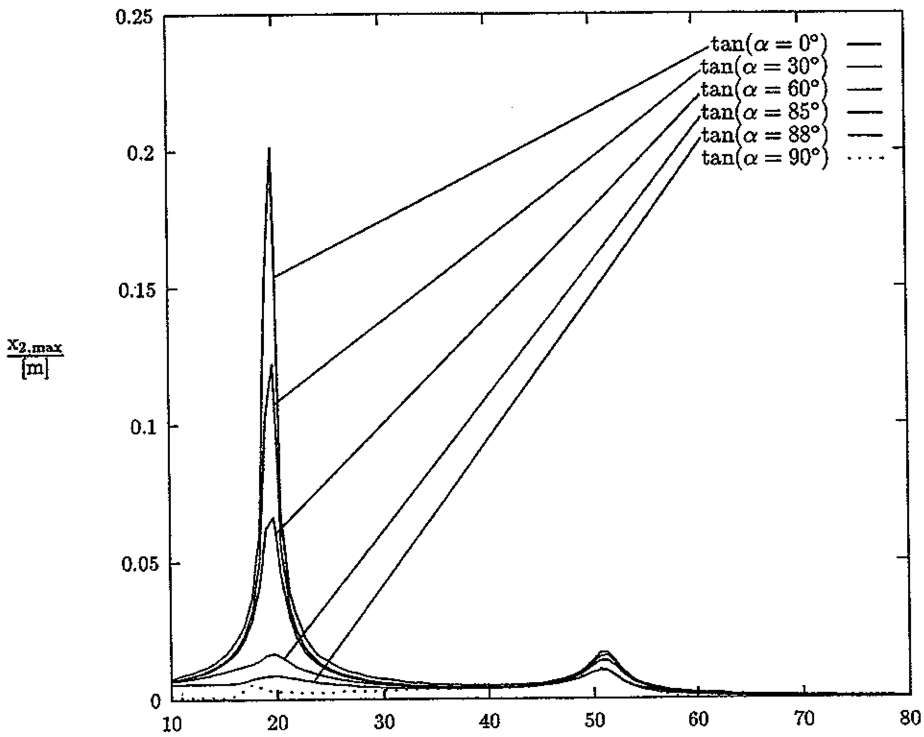

Figure 2 $x_{2,max}$ versus ν.

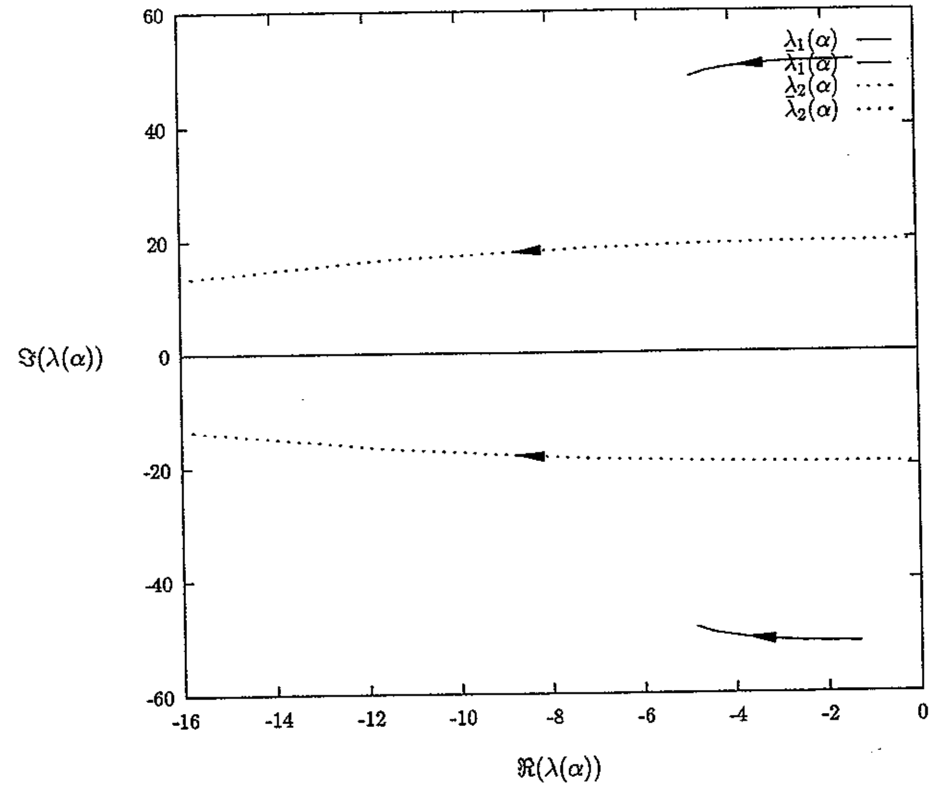

Figure 3 $\alpha \mapsto (\lambda_1, \lambda_2)$.

Figs. 1 and 2 show the amplitudes of state variables x_1 and x_2 versus the excitation frequency ν for different values of α. $\alpha = 0$ is equivalent to no control applied and $\alpha = \pi/2$ indicates the proposed Lyapunov approach. For $\alpha \longrightarrow \pi/2$ the figures also show that vibration attenuation improves significantly.

Fig. 3 shows the eigenvalues of $\tilde{A}(\alpha)$. As α tends to $\pi/2$ all real parts of the eigenvalues move towards more negative values. $Re\{\lambda_1\}$ has a strong tendency towards $-\infty$, while $Re\{\lambda_2\}$ seems to move to a value less than infinity. However, on the average, the tendency towards $-\infty$ appears to hold.

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